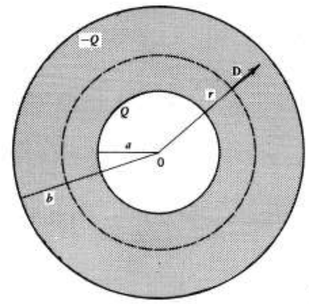
A spherical capacitor has two different layers of dielectrics between its plates. Their permittivities are ϵ_1 for a < r < r_0 and ϵ_2 for r_0 < r < b. Find the capacitance of this system by finding the total energy of the fields between the plates.

Let the charge on the innermost plate is Q. The field at distance r from the center of the spheres is given by

$$E = \frac{Q}{4\pi\epsilon_0\epsilon_r r^2}$$



Here ϵ_r is the dielectric constant at that point.

Now the volume density of electrostatic energy stored in an electric field is given by
Thus considering an infinitesimally thin spherical layer of radius r and thickness dr the energy stored in that layer is given by

$$dU = \frac{1}{2}\epsilon_0\epsilon_r E^2 dV = \frac{1}{2}\epsilon_0\epsilon_r E^2(4\pi r^2 * dr)$$

Or $$dU = \frac{1}{2}\epsilon_0\epsilon_r\left(\frac{Q}{4\pi\epsilon_0\epsilon_r r^2}\right)^2(4\pi r^2 * dr)$$

Or $$dU = \frac{Q^2}{8\pi\epsilon_0\epsilon_r} * \frac{1}{r^2} * dr$$

Thus energy stored in the capacitor can be calculated by integrating above equation for the two parts of different dielectrics and we get

$$U = \frac{Q^2}{8\pi\epsilon_0} * \left[\frac{1}{k_{e1}}\int_a^{r_0}\frac{dr}{r^2} + \frac{1}{k_{e2}}\int_{r_0}^{b}\frac{dr}{r^2}\right]$$

Or $$U = \frac{Q^2}{8\pi\epsilon_0}\left[\frac{1}{k_{e1}}\left(-\frac{1}{r_0}+\frac{1}{a}\right) + \frac{1}{k_{e2}}\left(-\frac{1}{b}+\frac{1}{r_0}\right)\right]$$

Now the energy stored in a capacitor is given by

$U = Q^2/2C$

Thus the equivalent capacitance of the given capacitor is given by

$$C = \frac{Q^2}{2U} = \frac{Q^2}{\frac{Q^2}{4\pi\epsilon_0}\left[\frac{1}{k_{e1}}\left(-\frac{1}{r_0}+\frac{1}{a}\right)+\frac{1}{k_{e2}}\left(-\frac{1}{b}+\frac{1}{r_0}\right)\right]}$$

Or $$C = \frac{Q^2}{2U} = \frac{4\pi\epsilon_0 k_{e1}k_{e2}r_0 ab}{[k_{e2}(r_0 b - ab) + k_{e1}(ab - r_0 a)]}$$

Or $$C = \frac{4\pi\epsilon_0 k_{e1}k_{e2}r_0 ab}{[(k_{e2}b - k_{e1}a)r_0 + ab\,(k_{e1} - k_{e2})]}$$