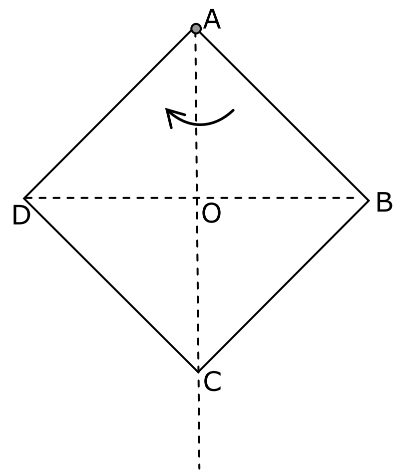
A uniform square lamina ABCD of mass m and side 2a is free to rotate about a horizontal axis thru A. The axis is perpendicular to the plane of lamina. The lamina is in equilibrium with C vertically below A when it is given an angular speed 2(g/a)^1/2 show that the lamina will perform complete revolutions and find the horizontal and vertical components of the reaction at A(a) when AC is horizontal (b) when C is vertically above A.

Moment of Inertia of a square lamina about an axis passing through its center (of mass) and normal to its surface

$$I_{CM} = \frac{1}{6}\left[m*(2a)^2\right] = \frac{2}{3}ma^2$$

Using parallel axis theorem, moment of inertia about an axis passing through one corner (A) and normal to surface will be

$$I = I_{CM} + md^2 = \frac{2}{3}ma^2 + m*\left(\sqrt{2}*a\right)^2 = \frac{8}{3}ma^2$$



The minimum energy required for completing full revolution will be equal the rise in potential energy for this position and will be equal to (rise in height of center of mass = 2*OA

$$\Delta U = mg\left(2\sqrt{2}*a\right) = 2.828mga$$

The kinetic energy given to the square is given by

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{8}{3}ma^2\right)\frac{4g}{a} = \frac{16}{3}mga = 5.33mga$$

As the kinetic energy imparted is greater than the maximum rise in potential energy the square plate will make complete revolution.

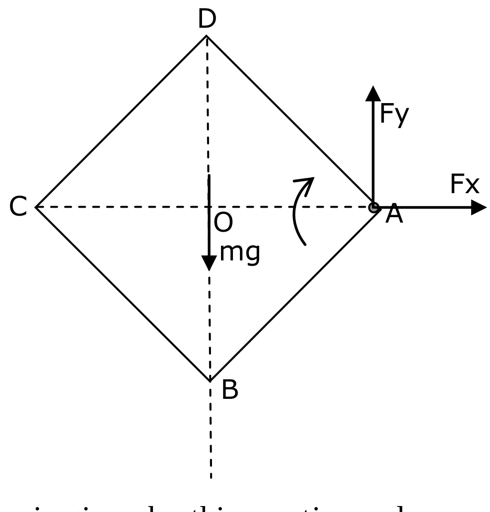
(a) When AC becomes horizontal, the angular velocity ω_1 is given by applying the law of conservation of energy as

Gain in PE = Loss in KE

Or $$\text{mg}\left(\text{a}\sqrt{2}\right) = \frac{16}{3}\text{mga} - \frac{1}{2}\text{I}\omega_1^2$$

Or $$\text{mg}\left(\text{a}\sqrt{2}\right) = \frac{16}{3}\text{mga} - \frac{1}{2}\left(\frac{8}{3}ma^2\right)\omega_1^2$$

Gives $$\omega_1^2 = \frac{3g}{4a}\left(\frac{16}{3} - \sqrt{2}\right) \qquad \text{---------- (1)}$$



As the center of mass of the lamina is moving on the circular path of radius a√2 and the horizontal reaction is the only force in this direction, the required centripetal force is given by this reaction only and hence we have

$$F_x = m\omega_1^2\left(a\sqrt{2}\right) = m\frac{3g}{4a}\left(\frac{16}{3} - \sqrt{2}\right)\left(a\sqrt{2}\right) = mg\left(4\sqrt{2} - \frac{3}{2}\right)$$

Now at this instant torque due to the weight about the axis of rotation will be

$$\tau = -mga\sqrt{2}$$

Hence angular acceleration at this instant will be

$$\alpha = \frac{\tau}{I} = -mga\sqrt{2} * \frac{3}{8ma^2} = -\frac{3g}{4\sqrt{2} * a}$$

And hence tangential acceleration of center of mass (vertical) at this instant will be

$$a = \alpha * a\sqrt{2} = -\frac{3g}{4}$$

Hence writing the equation of motion [F =ma] for center of mass at this instant we have

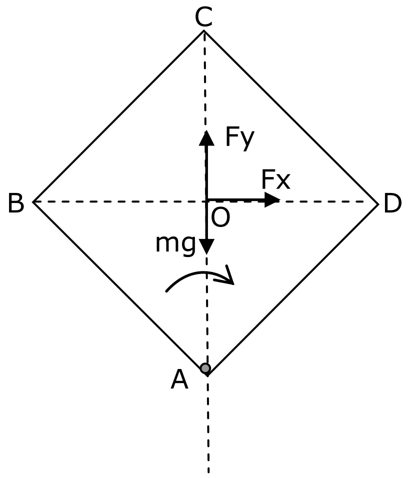
$$F_y - mg = m\left(-\frac{3g}{4}\right)$$

Or $$F_y = \frac{mg}{4}$$

(b)

When point C is vertically above A, the torque due to its weight mg about A will be zero and hence angular acceleration about A or linear acceleration of center mass in horizontal direction will be zero hence no force and so no component of reaction will be in horizontal direction will be there. Thus

$F_x' = 0$



Now the angular velocity at this instant is again given by the law of conservation of energy and hence

Gain in PE = loss in KE

Or $$mg\left(2\sqrt{2} * a\right) = \frac{16}{3}\text{mga-}\frac{1}{2}\text{I}\omega_2^2$$

Or $$\text{mg}\left(2\sqrt{2} * a\right) = \frac{16}{3}\text{mga-}\frac{1}{2}\left(\frac{8}{3}ma^2\right)\omega_2^2$$

Gives $$\omega_2^2 = \frac{3g}{4a}\left(\frac{16}{3} - 2\sqrt{2}\right)$$

The resultant of the reaction of the axis in vertical direction and the weight of the rod provides centripetal acceleration for center of mass at this instant hence writing equation on motion for this instant in vertical direction we have

$$F_y' - mg = -m\omega_2^2\left(a\sqrt{2}\right)$$

Or $$F_y' = mg - m * \frac{3g}{4a}\left(\frac{16}{3} - 2\sqrt{2}\right)\left(a\sqrt{2}\right) = mg\left[1 - 4\sqrt{2} + 3\right] = -4mg\left(\sqrt{2} - 1\right)$$

(Negative sign shows that the reaction on the lamina is vertically downward.)