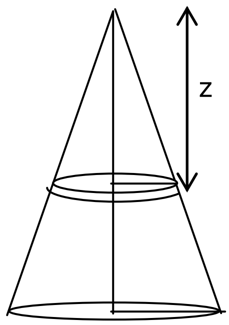
Q1- Find the center of mass of a uniform right circular solid cone of base radius a and height h.

The cone can be divided into infinite number of thin discs parallel to its base.

Consider one such disc element of thickness dz at a distance z from the vertex of the cone.



Radius of this disc element is given by solving the similar triangles as

$$\frac{r}{a}=\frac{z}{h}$$

Or $\quad$ r = (a*z)/h.

Hence volume of the disc is given by

V = area*thick = $\pi r^2 dz = \frac{\pi a^2 z^2}{h^2} * dz$

If ρ is the density of the material of the cone then mass dm of the disc will be

$$dm = \rho \frac{\pi a^2 z^2}{h^2} * dz$$

Now as we know that the center of mass of the disc is at its center the center of mass of the cone is given by

$$z_{CM} = \frac{\int zdm}{\int dm} = \frac{\int_0^h z * \rho \frac{\pi a^2 z^2}{h^2} * dz}{\int_0^h \rho \frac{\pi a^2 z^2}{h^2} * dz} = \frac{\int_0^h z^3 dz}{\int_0^h z^2 dz} = \frac{3}{4} h$$

Hence the center of mass of the cone is at a distance 3h/4 from the vertex of the cone or h/4 from the base, on the axis of the cone.

Q2- Find the center of mass of a hemispherical shell of constant density with inner radius r_1 and outer radius r_2.

The center of mass of a body is the point which moves linearly according to the laws of motion

$$\vec{F} = m\vec{a}$$

Irrespective of rotation and point of application of force

The position of center of mass for a system of particles is given by

$$\vec{r}_{cm} = \frac{\sum m\vec{r}}{\sum m}$$

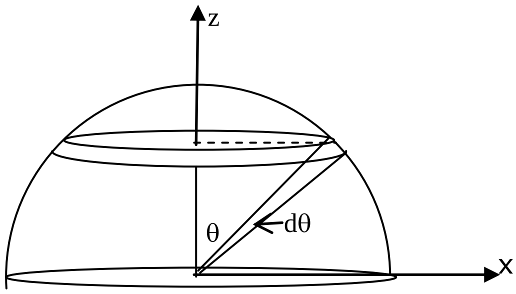
And for the mass distribution it is given by

$$\vec{r}_{cm} = \frac{\int \vec{r} dm}{\int dm}$$

Now for our shell we know that the mass is symmetrically distributed about its axis and hence its center of mass will be on the axis of symmetry.

The shell can be divided in a number of thin shells with different radii and hence first of all we will find the position of center of mass of a thin hemispherical shell of mass m and radius R.

Consider a thin hemispherical shell of mass m and radius R. this shell can be considered as made up of number of infinitesimally thin circular rings with their centers at the axis of shell. Consider one such ring subtending angle $d\theta$ at the center of the shell as in the figure.



The radius of this ring will be $R*\sin\theta$ and its width will be $R*d\theta$, hence the area of the ring will be

$$dA = 2\pi*R\sin\theta*R*d\theta = 2\pi R^2*\sin\theta*d\theta$$

Now as the mass of the shell is m and hence mass per unit area will be $m/(2\pi R^2)$, mass of the ring will be given by

$$dm = 2\pi R^2 \sin\theta * d\theta * \frac{m}{2\pi R^2} = m*\sin\theta*d\theta$$

Now as the center of mass of the ring is at its center, this mass may be considered on the z axis and at a distance $z = R\cos\theta$ from the center of the shell.

Hence as discussed above the center of mass of the shell is given by

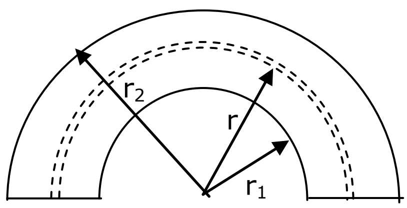
$$z_{cm} = \frac{\int z*dm}{\int dm} = \frac{\int_0^{\pi/2} R\cos\theta*m\sin\theta d\theta}{m} = R\int_0^{\pi/2}\cos\theta*\sin\theta d\theta = \frac{R}{2}$$

Now come to our hemispherical shell of density ρ and inner and outer radii r_1 and r_2 respectively. We may consider it as a combination of infinite thin shells.

Consider one such element shell having radius r and thickness dr. Mass of this shell element is given by

dM = volume*density = surface area*thickness*density

or $dM = 2\pi r^2 dr*\rho$



As we have already discussed, the center of mass of thin shell is at distance half of the radius from the center along axis the center of mass of this thin shell will be at $z = r/2$, and hence the center of mass of the given shell will be given by

$$z_{CM} = \frac{\int \frac{r}{2}*dM}{\int dM} = \frac{\int_{r1}^{r2}\frac{r}{2}*2\pi r^2 dr*\rho}{\int_{r1}^{r2} 2\pi r^2 dr*\rho} = \frac{\int_{r1}^{r2} r^3 dr}{2\int_{r1}^{r2} r^2 dr} = \frac{3}{8}*\frac{\left(r_2^4 - r_1^4\right)}{\left(r_2^3 - r_1^3\right)}$$

This is the distance of center of mass from the center of the hemispherical shell along the axis of the shell.