A raindrop is observed at time t=0 when it has mass m and downward velocity u. As it falls under gravity its mass increased by condensation at a rate k and a resisting force acts on it proportional to its speed and equal to kv when the speed is v. Show that $d(M^2 v)/dt = M^2 g$, where M = m +kt.

The rate of increase of mass is k kg per second.

At time t the mass of the raindrop will be m + k*t

Let the velocity of the raindrop at this moment (t) is v (taking downward direction as positive)

The net force acting on the raindrop at this moment (t) will be

F = (m +k*t)*g – k*v ------------------------- (1)

According to Newton's second law of motion, the rate of change of momentum is directly proportional to the force applied or for the variable mass system we have

$$F = \frac{dP}{dt} = \frac{d}{dt}\left(Mv\right) = M\frac{dv}{dt} + v\frac{dM}{dt}$$

Writing the equation of motion as above for the drop we have [dM/dt = k, given]

$$\left(m+kt\right)*g - kv = \left(m+kt\right)*\frac{dv}{dt} + vk$$

Or $$\left(m+kt\right)*\frac{dv}{dt} + 2kv = \left(m+kt\right)*g$$

Multiplying both sides by m + k*t we have

$$\left(m+kt\right)^2 * \frac{dv}{dt} + 2\left(m+kt\right)kv = \left(m+kt\right)^2 * g$$

Or $$\frac{d\left[\left(m+kt\right)^2 v\right]}{dt} = \left(m+kt\right)^2 * g$$

Or $$\frac{d\left[M^2 v\right]}{dt} = M^2 * g$$

Hence proved.

(m is the mass at t = 0, constant