Q- A bungee jumper of mass m is attached to one end of an elastic rope of length L and force constant k. The other end is fixed to a high bridge. The jumper steps off the bridge and falls from rest. If he just reaches the surface of river before going up again find

(a) The height of the bridge from the river.

(b) The maximum speed attained by the jumper during the drop.

(c) The time taken during the drop before coming to rest for the first time.

(a) As we know that at the jumper comes to rest at the surface of the river, applying law of conservation of energy we can write

Loss in gravitational potential energy = gain in elastic potential energy

Or $$mgh = \frac{1}{2}k(\Delta l)^2$$

Or $$mgh = \frac{1}{2}k(h-L)^2$$

Or $$2mgh = k(h^2 - 2Lh + L^2)$$

Or $$kh^2 - 2(kL+mg)h + kL^2 = 0$$

Or $$h = \frac{2(kL+mg) \pm \sqrt{4\,(kL+mg)^2 - 4k^2L^2}}{2k}$$

Or $$h = \frac{(kL+mg) \pm \sqrt{m^2g^2 + 2kLmg}}{k}$$

Or $$h = \frac{(kL+mg) \pm \sqrt{mg(2kL+mg)}}{k}$$

(Consider only positive root, other root is corresponding to the position after first rise)

(b) The speed will increase from zero to the point where acceleration becomes zero i.e. equilibrium position. If extension in the rope in this situation is x than kx = mg and using energy conservation rule

Loss in gravitational PE = gain in elastic PE + gain in KE

Or $$mg(L+x) = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

Substituting value of x we get

$$\frac{1}{2}mv^2 = mg\left(L + \frac{mg}{k}\right) = \frac{1}{2}k\left(\frac{mg}{k}\right)^2$$

Or $$v^2 = 2g\left(L + \frac{mg}{k}\right) - km\left(\frac{g}{k}\right)^2$$

Or $$v^2 = g\left(2L + \frac{mg}{k}\right)$$

Or $$v = \sqrt{g\left(2L + \frac{mg}{k}\right)}$$

(c) The time taken can be calculated in two parts, one for the motion with constant acceleration when the rope remains loose and the other for which the rope is taught and the force (acceleration) is proportional to the extension in the rope

For the first time interval using equation of motion $s = ut + \frac{1}{2}\, at^2$ we get

$$L = 0 + \frac{1}{2}\, gt_1^2$$

Or $$t_1 = \sqrt{\frac{2L}{g}}$$

For the second interval velocity changes from gt_1 to z

Let the velocity is v when the extension in the rope is x, than using conservation of energy we get

$$\frac{1}{2} mv^2 = mg(L + x) - \frac{1}{2}\, k\, x^2$$

Or $$v = \sqrt{2g(L + x) - \, (k/m)\, x^2}$$

Or $$\frac{dx}{dt} = \sqrt{2g(L + x) - \, (k/m)\, x^2}$$

Or $$\frac{dx}{\sqrt{2g(L+x) - (k/m)\, x^2}} = dt$$

Or $$t_2 = \int_0^{h-L} \frac{dx}{\sqrt{2g(L+x) - (k/m)\, x^2}}$$

Or $$t_2 = \int_0^{h-L} \frac{dx}{\sqrt{2gL + 2gx - (k/m)\, x^2}}$$

Or $$t_2 = \int_0^{h-L} \frac{dx}{\sqrt{2gL + \frac{mg^2}{k} - \frac{mg^2}{k} + 2gx - (k/m)\, x^2}}$$

Or $$t_2 = \int_0^{h-L} \frac{dx}{\sqrt{2gL + \frac{mg^2}{k} - \left(\sqrt{\frac{k}{m}}\, x - \sqrt{\frac{m}{k}}\, g\right)^2}}$$

Or $$t_2 = \sqrt{\frac{m}{k}} * \left| \sin^{-1}\left(\frac{\sqrt{\frac{k}{m}}\, x - \sqrt{\frac{m}{k}}\, g}{\sqrt{2gL + \frac{mg^2}{k}}}\right)\right|_0^{h-L}$$

Or $$t_2 = \sqrt{\frac{m}{k}} * \left[\sin^{-1}\left(\frac{\sqrt{\frac{k}{m}}\,(h-L) - \sqrt{\frac{m}{k}}\, g}{\sqrt{2gL + \frac{mg^2}{k}}}\right) + \sin^{-1}\left(\frac{\sqrt{\frac{m}{k}}\, g}{\sqrt{2gL + \frac{mg^2}{k}}}\right)\right]$$

Thus total time to reach the surface will be

$$t = t_1 + t_2$$

Or

$$= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}} * \left[\sin^{-1}\left(\frac{\sqrt{\frac{k}{m}}\,(h-L) - \sqrt{\frac{m}{k}}\, g}{\sqrt{2gL + \frac{mg^2}{k}}}\right) + \sin^{-1}\left(\frac{\sqrt{\frac{m}{k}}\, g}{\sqrt{2gL + \frac{mg^2}{k}}}\right)\right]$$