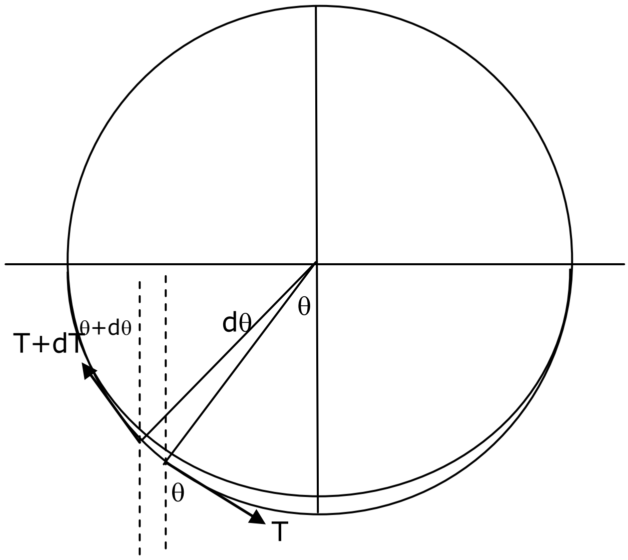
Q- A rubber band of spring constant K is tightly wrapped (without tension) around a metal rod of diameter D. The rod is then accelerated upward with acceleration 100g. What will be the gap between the metal rod and the rubber band at the bottom? Mass per unit length of the rubber band is m.

Due to the motion of the metal rod the upper half part of the rubber band is subjected to the normal force and friction force and will move with the rod with equal acceleration 100 g (I think it is the acceleration due to gravity denoted by small g not G)

The lower half will have to accelerate with the same acceleration but there is no upward force hence a tension is developed in each part of the band and that too is varying. This tension will accelerate the lower part, as well as stretch the band to increase its length and there will be a gap in the band and the rod. Now if the change in length of the lower part is small, it can be still considered semicircular.



Now consider an infinitesimally small part of the band subtending angle between θ and θ + d θ as shown in diagram.
Let the tension on the two ends are T and T + dT
Now these tensions can be resolved in horizontal and vertical direction
[For small angles sin dθ = dθ and cos dθ = 1]

The horizontal component of the net force on the element will be

$$F_x = T\cos\theta - (T + dT)\cos(\theta + d\theta)$$

Or $$F_x = T\cos\theta - (T + dT)(\cos\theta * 1 - \sin\theta * d\theta)$$

Or $$F_x = -dT\cos\theta + (T + dT)\ \sin\theta * d\theta$$

Or $$F_x = -dT\cos\theta + T\sin\theta * d\theta + dT\sin\theta * d\theta$$

Or $$F_x = -dT\cos\theta + T\sin\theta * d\theta$$ [$dT * d\theta$ is negligibally small]

As the element is not moving horizontally the net horizontal force must be zero thus

$$F_x = -dT\cos\theta + T\sin\theta * d\theta = 0$$

Gives dT = T*tanθ*dθ ---------------- (1)

The length of this element is given by ρ = (D/2) dθ, and as the mass per unit length is m the mass of element will be m (D/2) dθ.

Now the vertical component of the all forces sums up to

$$F_y = (T + dT)\sin(\theta + d\theta) - T\sin\theta - m * \frac{D}{2} * d\theta * g$$

Or $$F_y = (T + dT)(\sin\theta\cos d\theta + \cos\theta\sin d\theta) - T\sin\theta - m * \frac{D}{2} * d\theta * g$$

Or $$F_y = (T + dT)(\sin\theta * 1 + \cos\theta * d\theta) - T\sin\theta - m * \frac{D}{2} * d\theta * g$$

Or $$F_y = T\cos\theta * d\theta + dT\sin\theta + \ dT\cos\theta\ d\theta - m * \frac{D}{2} * d\theta * g$$

Or $$F_y = T\cos\theta * d\theta + dT\sin\theta - m * \frac{D}{2} * d\theta * g$$ [$dT * d\theta$ is negligibally small]

We can write its equation of motion for vertical direction as

[F = ma]

$$F_y = T\cos\theta * d\theta + dT\sin\theta - m * \frac{D}{2} * d\theta * g = m * \frac{D}{2} * d\theta * 100g$$

Or $$T\cos\theta * d\theta + dT\sin\theta = m * \frac{D}{2} * d\theta * 101g$$

Substituting the value of dT from equation (1) we get

$$T\cos\theta * d\theta + T\tan\theta * d\theta * \sin\theta = m * \frac{D}{2} * d\theta * 101g$$

Or $$T(\cos\theta + \tan\theta * \sin\theta) = \frac{101}{2}\, mDg$$

Or $$T = \frac{101}{2}\, mDg\cos\theta \qquad \text{------------------------- (2)}$$

Now the spring constant for the whole band (length πD) is K

Hence the force constant of length ρ = (D/2)dθ of the band element will be given by

$$K' = \frac{\pi DK}{\left(\frac{D}{2}\right)d\theta} = \frac{2\pi K}{d\theta}$$

Hence increase in the length of the element is given by

$$d\rho = \frac{T}{K'} = \frac{T * d\theta}{2\pi K}$$

Substituting the value of T from equation (2) the increase in the length of the element is given by

$$d\rho = \frac{101}{2}\frac{mDg}{2\pi K}\cos\theta\ d\theta \qquad \text{------------------------- (3)}$$

This increase in the length is along the length of the element and hence the component of this increase in vertical direction will be given by

dh = dρ*sin θ

As these small increments in the different elements will increase the gap between the rod and the band the total gap between the lower part of the band and the rod is given by

$$h = \int dh = \int d\rho * \sin\theta$$

Substituting the value from equation (3) and integrating between the proper limits we have

$$h = \int_0^{\frac{\pi}{2}} \frac{101}{2}\frac{mDg}{2\pi K}\sin\theta\cos\theta\ d\theta$$

or $$h = \frac{101mDg}{4\pi K}\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\ d\theta$$

or $$h = \frac{101mDg}{8\pi K}\int_0^{\frac{\pi}{2}} \sin 2\theta\ d\theta$$

or $$h = -\frac{101mDg}{8\pi K}\left[\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

or $$h = -\frac{101mDg}{16\pi K}[\cos\pi - \cos 0]$$

or $$h = \frac{101mDg}{8\pi K}$$

This is the gap in the band and the rod at the lowest point.

This expression is correct for small increments only, If the spring constant of the band is very low it will not be exact.