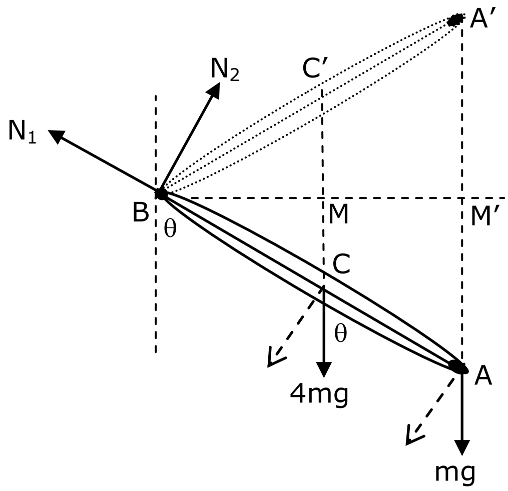
Q- A uniform circular disc has mass 4m and radius r. A particle of mass m is attached to the disc at point A of its circumference. The loaded disc is free to rotate about a horizontal axis which is tangential to the disc at the point B, where AB is a diameter. The disc is released from rest with AB at an angle of 60 degree with the upward vertical. Find the magnitude of the force exerted by the disc 0n the axis when AB makes an angle 60 with the downward vertical.



The moment of inertia of a disc about its diameter is

I_d = ¼ mass*radius^2

Or I_d = ¼*(4m) r^2 = mr^2

Hence the moment of inertia about an axis parallel and passing through the edge (means tangential) is given by the parallel axis theorem as

I_B = I_d + mass*distance^2

Or I_B = mr^2 + (4m)*r^2 = $5mr^2$

This is the moment of inertia of the disc about B.

As the distance of the particle of mass m from the axis B is 2r, its moment of inertia about it will be m*$(2r)^2$ = $4mr^2$

Thus the moment of inertia of the disc with the particle about axis through B will be

I = $5mr^2$ + $4mr^2$ = $9mr^2$ ----- (1)

Now as the diameter of the disc makes an angle of 60^0 in both positions with vertical hence triangle BA′A will be an equilateral triangle and thus A′A will be 2r and C′C will be r.

As there no non-conservative force acting on the system, according to law of conservation of energy

Loss in Potential energy = gain in kinetic energy

Loss in PE (of the disc + of the particle) = gain in rotational KE of the system

Or 4mg*CC′ + mg*AA′ = ½*I*ω^2

Or 4mg*r + mg*2r = ½*9mr^2*ω^2

Where ω is the angular velocity of the disc in 2nd position AB

Gives ω^2 = 4g/3r. ----------- (2)

The position of center of mass on the disc and the particle from the axis of rotation AB is given by

r_{CM} = $\dfrac{4m*r+m*2r}{4m+m}=\dfrac{6r}{5}$ ------------- (3)

The torque acting on the disc in 2nd position is given by (force*perpendicular distance)

τ = 4mg*MB + mg*M′B = 4mg*r sin60^0 + mg*2r sin60^0 = 3√3*mgr

Hence angular acceleration of the system in position AB will be

$$\alpha = \frac{\tau}{I} = \frac{3\sqrt{3} * mgr}{9mr^2} = \frac{g}{\sqrt{3} * r}$$

And thus the acceleration of center of mass of the system (perpendicular to AB) will be given by

$$a = \alpha * r_{CM} = \frac{g}{\sqrt{3} * r} * \frac{6r}{5} = \frac{2\sqrt{3} * g}{5} \qquad \text{-------------- (4)}$$

Now the external forces acting on the system of the disc and particle are

1. the weight of the disc 4mg
2. the weight of the particle mg and
3. the reaction of the axis N

Resolving the forces in the direction AB and perpendicular to it we have

(A) The force in the direction AB

$$F_{AB} = N_1 - 4mg\cos 60^0 - mg\cos 60^0 = N_1 - (5mg/2)$$

This force provides the required centripetal force to move the mass along the circular path and hence

$$F_{AB} = N_1 - (5mg/2) = 5m * \omega^2 * r_{CM}$$

Substituting the values of ω^2 and r_{CM} from equations 2 and 3 and rearranging we have

$$N_1 = \frac{5mg}{2} + 5m * \frac{4g}{3r} * \frac{6r}{5} = \frac{5mg}{2} + 8mg = \frac{21mg}{2} \qquad \text{---------------- (A)}$$

(B) The force in the direction perpendicular to AB will be

$$F_{PER} = 4mg\sin 60^0 + mg\sin 60^0 - N_2 = \frac{5\sqrt{3}}{2} * mg - N_2$$

This force will provide the acceleration to the mass of the system on the circular path and hence we have

$$F_{PER} = \frac{5\sqrt{3}}{2} * mg - N_2 = 5m * a$$

Substituting the value of a from equation 4 and rearranging we have

$$N_2 = \frac{5\sqrt{3}}{2} * mg - 5m * \frac{2\sqrt{3} * g}{5} = \frac{\sqrt{3} * mg}{2}$$

Hence the reaction of the axis on the disc is given by

$$N = \sqrt{N_1^2 + N_1^2} = \sqrt{\left(\frac{21mg}{2}\right)^2 + \left(\frac{\sqrt{3}mg}{2}\right)^2} = \sqrt{\frac{441}{4} + \frac{3}{4}}mg = \sqrt{111} * mg$$

As according to Newton's third law of motion every force have equal and opposite reaction, the force exerted by the disk will also have a magnitude equal to

F = mg$\sqrt{(111)}$.