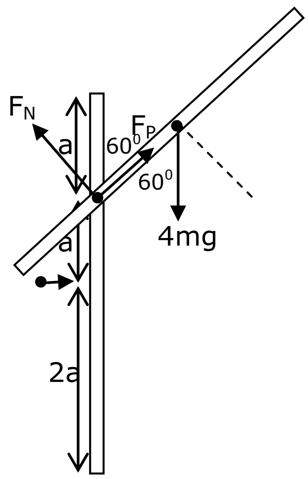
Q- A uniform rod AB, of mass m and length 4a, is free to rotate in a vertical plane about a fixed horizontal axis through the point X on the rod, where AX=a. The rod is hanging at rest with B below A, when it is struck at its mid point by a particle of mass 3m moving horizontally with speed u in a direction perpendicular to the rod. Immediately after the impact P adheres to the rod. When the rod comes to instantaneous rest, the force exerted by the axis on the rod is F.

(a)Show that after the impact, the moment of inertia about the axis of the rod and particle together is $16/3ma^2$

The moment of inertia of a rod of mass m and length L about an axis perpendicular to its length and passing through its center of mass is $mL^2/12$ hence the moment of inertia of the rod about its axis of rotation at a distance a from its center of mass will be given by parallel axis theorem as

$$I_R = m*(4a)^2/12 + ma^2 = (28ma^2)/12 = 7ma^2/3$$



Moment or inertia of the particle about the axis is

$$I_P = (3m)*a^2$$

Hence the moment of inertia of the system about the rotation axis will be

$$I = I_R + I_P = (7ma^2)/3 + (3m)*a^2$$

$$= (16ma^2)/3$$

(b) Find in terms of u and a, the angular speed of the rod immediately after the impact.

As the particle of mass 3m is moving with velocity u horizontally at a distance a from the axis of rotation, the angular momentum of the particle about the axis is given as

L = moment of momentum
= momentum* perpendicular distance from axis
= 3mu*a

Now if the angular speed of the system just after impact is ω, than its angular momentum will be

$= I*\omega$

As there is no external torque at that moment, applying law of conservation of angular momentum we get

$L = I\,\omega = 3mua$

Or $\omega = 3mu/I = 3mua*3/(16ma^2) = 9u/16a$

In the subsequent motion, the rod comes to instantaneous rest when it has rotated through an angle of 120^0

(c) Find the velocity of the particle u. Show that u=8/3(ag)^1/2

Now the kinetic energy of the system just after impact is

$KE = ½\, I\omega^2 = ½ *(16ma^2/3)*(9u/16a)^2 = 27mu^2/32$

The increase in potential energy of the system when it comes to instantaneous rest will be

Increase in PE = that of the rod + that of the particle

Or $DPE = mg(a + a\cos 60^0) + 3\,mg(a + a\cos 60^0) = 4mg*3a/2 = 6\,mga$

Now according to law of conservation of energy we get

Loss in KE = gain in PE

Or $27mu^2/32 = 6\ mga$

Gives $u = \frac{8}{3}\sqrt{ga}$

Find in terms of m and g, the magnitude of the component F which is

(d) parallel to the rod

Now the forces acting on the rod when its velocity is zero are the weight of the rod and the particle 4m acting at midpoint of the rod and the reaction of the axis N.

Resolving these in the direction parallel to the rod, the components are $4mg\cos 60^0$, and F_P and as the center of mass of the system is neither moving in this direction nor it is having any acceleration in this direction the forces are balancing each other and we have

$F_P - 4mg\cos 60^0 = 0$

[As the angular velocity of the rod is zero there will be no centripetal acceleration]

Or $F_P = 4mg\cos 60^0$

Gives $\mathbf{F_P = 2mg}$

(e) Perpendicular to the rod

The torque acting on the rod due to the mass of the rod and particle will be

Torque = force*perpendicular distance from the axis of rotation

Or $\tau = 4mg*a*\sin 60^0 = 2\sqrt{3}\ mg*a$

Hence the angular acceleration of the system when it is instantaneously at rest is given by

$$\alpha = \frac{\tau}{I} = \frac{2\sqrt{3}*mg*a}{16ma^2/3} = \frac{3\sqrt{3}*mga}{8ma^2} = \frac{3\sqrt{3}*g}{8a}$$

Hence the acceleration of center of mass (perpendicular to the rod) will be

$$f = \alpha * a = \frac{3\sqrt{3}*g}{8} \qquad \text{---------------------} \ (1)$$

Now writing the equation of motion [F = ma] in the direction perpendicular to the rod for the system we have

$$4mg*\sin 60^0 - F_N = 4m*f$$

Or $F_N = 4mg*\sin 60^0 - 4m*f$

Substituting the value of acceleration from equation (1) we get

$$F_N = 4mg*\frac{\sqrt{3}}{2} - 4m*\frac{3\sqrt{3}*g}{8}$$

Or $F_N = 2\sqrt{3}*mg - \frac{3\sqrt{3}}{2}*mg$

or $F_N = \frac{7\sqrt{3}}{2}*mg$