Q- A cylinder of hydrogen, 2m in height and with a 25cm diameter, is dropped and develops a crack of size 0.1 mm^2. The gas is stored at 10 bar absolute pressure. Estimate how long it takes for half the contents to leak out. Explain your reasoning and assumptions.

Solution:
Volume of cylinder V = 2*3.14*$(0.125)^2$ = 0.098 m^3
Initial pressure Pin = 10 bar = 10*10^5 Pa, Atm. pressure P_0 = 1.013*10^5 Pa
Area of the crack A = 0.1 mm^2 = 1.0*10^{-7} m^2; density of hydrogen ρ = 0.0899 kg/m^3

Assumptions

1. As nothing is told about temperature and the insulation, we assume the gas temperature is equal to the atmospheric temperature and remains unchanged. Thus the process of leaking of the gas may be considered isothermal.

2. As coefficient of viscosity is not given the flow of the gas is non viscous flow.

The gas will flow out of the cylinder because of the pressure difference inside and outside it. As the gas leaks out with time, the quantity of gas and hence the pressure of the gas in the cylinder decreases. The initial pressure Pi is 10 bar thus the pressure of the gas when the content becomes half will be Pf = Pi/2 = 5 bar.

Let at time t the pressure of the gas is P and the atmospheric pressure is P_0 then the velocity of exit flow of the gas v is given by using the Bernoulli equation (Just before expansion) as $\frac{1}{2}\rho v^2 = P - P_0$

Gives $v = \sqrt{\frac{2(P - P_0)}{\rho}}$ --------------------------- (1)

Here ρ is the density of gas.

Now let the number of moles of the gas at time t be n and in an infinitely small interval of time dt, dn moles flows out and rest of gas occupies the whole volume V of the gas. The pressure of the gas inside reduces from P to P – dP. Thus from the gas law we get

PV = n RT ----------- (A) and (P - dP) V = (n – dn) RT ------- (B)

Subtracting (B) from (A) and dividing by (A) we get

$\frac{dn}{n} = \frac{dP}{P}$ --------------------------------- (2)

The volume dV of the dn moles of the gas outside of the cylinder (after expanding) is given by

P_0*dV = dn RT = dn PV/n (using ideal gas equation)

Or $dV = \frac{dn}{n} * \frac{P}{P_0} * V$

Using equation (2) we get

$$dV = \frac{dP}{P} * \frac{P}{P_0} * V = \frac{dP}{P_0} * V \qquad \text{--------------------------------- (3)}$$

Now as the area of the crack is A and the flow velocity of the gas is v the volume flow rate is given by

$$\frac{dV}{dt} = A * v = A * \sqrt{\frac{2(P - P_0)}{\rho}}$$ Or

$$dV = A * \sqrt{\frac{2(P - P_0)}{\rho}}\ dt$$

substituting dV from equation (3) we get

$$\frac{dP}{P_0} * V = A * \sqrt{\frac{2(P - P_0)}{\rho}}\ dt$$

Or $$\frac{dP}{\sqrt{P - P_0}} = \frac{A\,P_0}{V} \sqrt{\frac{2}{\rho}} *\ dt$$

Integrating for the final pressure $P_0/2$ in time t we get

$$\int_{P_{in}}^{P_{in}/2} \frac{dP}{\sqrt{P - P_0}} = -\frac{A\,P_0}{V} \sqrt{\frac{2}{\rho}} * \int_0^t dt$$

(- ve sign, because the pressure decreases with increase in time)

Or $$2\left[\sqrt{P - P_0}\right]_{P_{in}}^{P_{in}/2} = -\frac{A\,P_0}{V} \sqrt{\frac{2}{\rho}} * t$$

Or $$2\left[\sqrt{P_{in} - P_0} - \sqrt{\frac{P_{in}}{2} - P_0}\right] = \frac{A\,P_0}{V} \sqrt{\frac{2}{\rho}} * t$$

Gives $$t = \frac{V}{A} \frac{\sqrt{2\,\rho}}{P_0} \left[\sqrt{P_{in} - P_0} - \sqrt{\frac{P_{in}}{2} - P_0}\right]$$

Or $$t = \frac{0.098}{10^{-7}} \frac{\sqrt{2 * 0.0899}}{1.013 * 10^5} \left[\sqrt{10^6 - 1.013 * 10^5} - \sqrt{\frac{10^6}{2} - 1.013 * 10^5}\right]$$

Or $$t = 4.1\,[948.0 - 631.4] = 1298\ s = \mathbf{21.63\ min}$$