Q- a small body of mass m_o is projected vertically upwards in a cloud. Its initial speed is $\sqrt{2gk}$. During its motion the body picks up moisture from the stationary cloud. Its mass at height x above the point of projection is $m_0(1+\rho x)$, where ρ is a +ve constant. Show that the greatest height h satisfies the equation
$(1+\rho h)^3=(1+3k\rho)$

According to Newton's second law of motion we know

$$\frac{dP}{dt}=F$$

Or $$\frac{dP}{dt}*\frac{dx}{dx}=\frac{dP}{dx}*\frac{dx}{dt}=\frac{dP}{dx}*v=F$$

The force acting on the body at the instant when its mass is m, is the gravitational force mg, (downwards) we can write its equation of motion as

$$v*\frac{d}{dx}(mv)=-mg$$

or $$mv\frac{dv}{dx}+v^2\frac{dm}{dx}=-mg$$

When the height of the body is x its mass m = $m_0(1+\rho x)$ hence substituting the value we have

$$m_0(1+\rho x)v\frac{dv}{dx}+v^2\frac{dm_0(1+\rho x)}{dx}=-m_0(1+\rho x)g$$

or $$(1+\rho x)v\frac{dv}{dx}+v^2\frac{d(1+\rho x)}{dx}=-(1+\rho x)g$$

or $$2v\frac{dv}{dx}+v^2\frac{2\rho}{(1+\rho x)}=-2g$$

or $$\frac{d(v^2)}{dx}+v^2\frac{2\rho}{(1+\rho x)}=-2g \qquad \text{----------------------------- (1)}$$

This is a linear differential equation in v^2
Now

$$\int\frac{2\rho}{1+\rho x}dx=2\ln(1+\rho x)$$

Hence integrating factor, I = $\int\frac{2\rho}{1+\rho x}dx=2\ln(1+\rho x)=(1+\rho x)^2$

Multiplying the equation (1) by integrating factor I we have

$$(1+\rho x)^2\frac{d(v^2)}{dx}+v^2 2\rho(1+\rho x)=-2g(1+\rho x)^2$$

Or $$\frac{d}{dx}\left[v^2(1+\rho x)^2\right]=-2g(1+\rho x)^2$$

Integrating above equation with respect to x we get

$$\frac{d}{dx}\left[v^2(1+\rho x)^2\right]=-2g\int(1+\rho x)^2dx+C$$

or $$\left[v^2(1+\rho x)^2\right]=-2g\frac{(1+\rho x)^3}{3\rho}+C \qquad \text{----------------------------- (2)}$$

C is the constant of integration.

Initially at x = 0 the initial velocity v_0 is given as $v_0 = \sqrt{(2gk)}$, substituting these values in equation (2) we get

$$\left[2gk*(1+\rho*0)^2\right] = -2g\frac{(1+\rho*0)^3}{3\rho} + C$$

Gives $C = 2gk + \dfrac{2g}{3\rho}$

Substituting this value of C in equation (2) we get the velocity of the body as a function of its height x as

$$\left[v^2(1+\rho x)^2\right] = -2g\frac{(1+\rho x)^3}{3\rho} + 2gk + \frac{2g}{3\rho} \quad \text{------------------} \ (3)$$

Now when the height of the body h is the greatest its velocity will be zero and hence substituting greatest height h for x and v = 0 in equation (3) we get

$$0 = -2g\frac{(1+\rho h)^3}{3\rho} + 2gk + \frac{2g}{3\rho}$$

Or $2g\dfrac{(1+\rho h)^3}{3\rho} = 2gk + \dfrac{2g}{3\rho}$

Or $(1+\rho h)^3 = 3\rho k + 1$

Hence shown