Q- Calculate the work done on a 1000 square meter, 1 kg solar sail during a trip from Earth to Mars. Assume the sunlight is normal to the sail during the trip and is totally reflective. Consider forces of gravity of the sun and its radiation pressure only.

Reading:

Radiation pressure is the pressure exerted on a surface due to impact on it by the incident photon. The momentum of the photon is absorbed or reflected by the surface accordingly and as the rate of change of momentum is the force applied, the surface experiences a force.

Using Einstein's theory momentum of photons (mass less) is given by

$$p = E/c$$

Here E is the energy of photons and c is speed of light.

Now as the power of the sun (energy emitted per unit time) is dW/dt ($3.9*10^{26}$ W) the intensity (amount of energy incident per unit area per unit time) of the waves at distance 'r' from the sun is given by

$$I = \frac{1}{4\pi r^2} * \frac{dW}{dt}$$

Hence power incident per unit time on the surface of the sail of area A will be

$$\frac{dE}{dt} = IA = \frac{A}{4\pi r^2} * \frac{dW}{dt}$$

Hence momentum incident per unit time on the surface will be (normal incident)

$$\frac{dp}{dt} = \frac{1}{c}\frac{dE}{dt} = \frac{1}{c}\frac{A}{4\pi r^2} * \frac{dW}{dt}$$

But as the photons are reflected at the same speed the net rate of change of momentum of the photon beam will be

$$p_f - p_{in} = \left(-\frac{dp}{dt}\right) - \frac{dp}{dt} = -\frac{2}{c}\frac{A}{4\pi r^2} * \frac{dW}{dt}$$

Hence according to Newton's second law rate of change of momentum i.e. force experienced by the (surface) body $= \frac{2}{c}\frac{A}{4\pi r^2} * \frac{dW}{dt}$

There are only two forces are to be considered. Force due to radiation pressure, in the direction of r, and the gravity force of the sun in the direction –r (attractive). Thus net force acting on the solar sail is given by

$$\vec{F} = \vec{F}_{radiation} + \vec{F}_{gravity}$$

Or $$\vec{F} = \frac{2}{c}\frac{A}{4\pi r^2} * \frac{dW}{dt}\,\hat{r} + \frac{GM_S m}{r^2}\,(-\hat{r})$$

Or $\vec{F} = \left[\frac{2}{c}\frac{A}{4\pi} * \frac{dW}{dt} - GM_s m\right]\frac{\hat{r}}{r^2}$

Thus the work done on the sail is given by

$$Work = \int \vec{F}.d\vec{r} = \int_{r_{SE}}^{r_{SM}} F\,dr = \left[\frac{2}{c}\frac{A}{4\pi} * \frac{dW}{dt} - GM_s m\right]\int_{r_{SE}}^{r_{SM}}\frac{dr}{r^2}$$

Or $Work = \left[\frac{2}{c}\frac{A}{4\pi} * \frac{dW}{dt} - GM_s m\right]\left[\frac{1}{r_{SE}} - \frac{1}{r_{SM}}\right]$

Substituting numerical values given in part a and b of the questions we get

$$Work = \left[\frac{2}{3*10^8}\frac{1000}{4*3.14} * 3.9 * 10^{26} - 6.7 * 10^{-11} * 2 * 10^{30} * 1\right]\left[\frac{1}{1.5*10^{11}} - \frac{1}{2.3*10^{11}}\right]$$

Or $Work = [2.07 * 10^{20} - 1.34 * 10^{20}]\,[2.32 * 10^{-12}] = \mathbf{1.69 * 10^8\,J}$