Q- A thin rod is held vertically with its lower end resting on a frictionless horizontal surface. The rod is then released to fall freely.

(a) Determine the speed of its center of mass just before it hits the horizontal surface. Use h, M, and g for length, mass, and gravitational acceleration respectively.

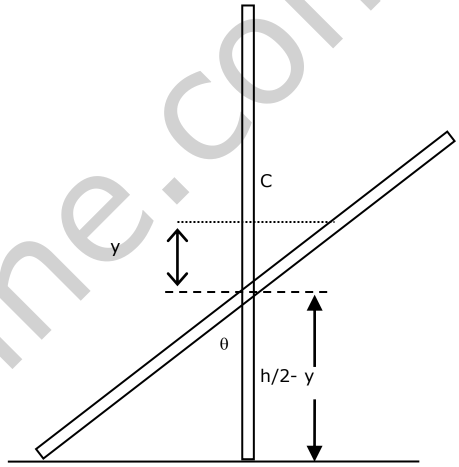
(b) Now suppose the rod has a fixed pivot at its lower end, determine the speed of the rod's center of mass just before it hits the surface.

Answer:

(a)

1. As there is no friction between the rod and the horizontal surface, no external force acting on the rod in horizontal direction during the motion and the center of mass of the rod moves straight downwards.

2. As there is no non-conservative force acting on the rod and hence the total energy of the rod remains conserved.



The end of the rod touching the surface slides backward. The center of mass of the rod moves vertically and the rod rotates about the axis passing through center of mass.

Let at time t = 0 the rod is released and at time t it is making angle θ with the vertical and its center of mass is at a distance y lower than its initial position.

Height of center of mass initially h/2 and at time t is h/2 –y, hence loss in potential energy is

$$mg\,(h/2 - h/2 + y) = mgy$$

The total kinetic energy at time t is rotational + transnational kinetic energy which is given by

$$(1/2)\,m\,(dy/dt)^2 + (1/2)\,I_C(d\theta/dt)^2 = (1/2)m(dy/dt)^2 + (mh^2/24)(d\theta/dt)^2$$

where $I_C = mh^2/12$ is the moment of inertia of the rod about its center of mass.

According to the law of conservation of energy

Loss in P.E. = gain in K.E.

Or $$mgy = (1/2)m(dy/dt)^2 + (mh^2/24)(d\theta/dt)^2 \qquad \text{........(1)}$$

Now we can relate θ and y through the triangle as

$$h/2 - y = (h/2)\cos\theta$$

or $$y = (h/2)(1-\cos\theta)$$

gives by differentiating w.r.t. t

$$dy/dt = v_y = (h/2)(0 + \sin\theta)\,(d\theta/dt)$$

or $$v_y = (h\sin\theta)\,(d\theta/dt)/2 \qquad \text{........(2)}$$

Substituting value of (dθ/dt) in equation (1) and (2) we get

$$mgy = (1/2)m(v_y)^2 + (mh^2/24)(2v_y/h \sin\theta)^2$$

or $$v_y^2 = 6gy\sin^2\theta/(3\sin^2\theta + 1)$$

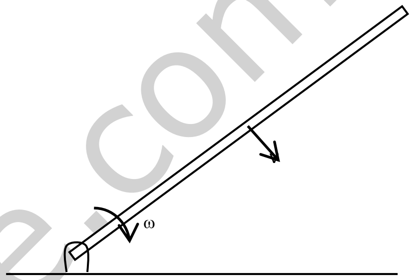
or $$v_y = \sqrt{[6gy\sin^2\theta/(3\sin^2\theta + 1)]}$$

Now just before hitting the surface $y = h/2$ and $\theta = 90^0$. Substituting these values in the equation (3)

$$v_y = \sqrt{[6gy\sin^2\theta/(3\sin^2\theta + 1)]}$$

or $$v_y = \mathbf{\sqrt{[3gh/4]}}$$

(b) If the rod is hinged at the lower end then the rod will rotate about the lower end due to the torque produced by the weight mg of the rod. Still no non-conservative force is acting and hence the energy will be conserved. The motion can be considered only rotational about the axis passing through the lower end.



According to law of conservation of energy

Loss in potential energy = gain rotational kinetic energy

$$mgh/2 = (1/2)I_E\omega^2.$$

Where ω is angular velocity of the rod and I_E is the <u>moment of inertia of the rod about the lower end.</u>

I_E can be given by using parallel axis theorem as

$$I_E = I_C + m(h/2)^2 = (mh^2/12) + m(h/2)^2 = mh^2/3$$

Hence

$$mgh/2 = (1/2)(mh^2/3)\omega^2 = mh^2\omega^2/6$$

or $$\omega = \sqrt{(3g/h)}$$

hence the linear velocity of center of mass $v_y = (h/2) = (h/2)\sqrt{(3g/h)} = \mathbf{\sqrt{(3gh/4)}}$

note: in both cases the velocity of the center of mass just before hitting is the same because in the first case too, like the second, the lower end comes to rest and the final motion is same except the position of center of mass of the rod.