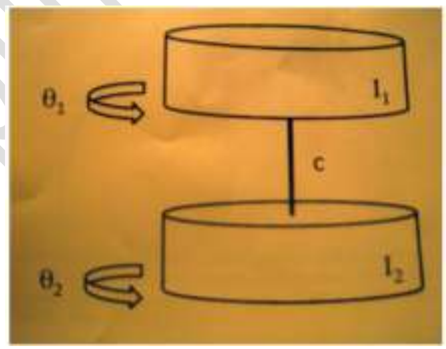
Q- Two disks with moments of inertia I_1 (top disk) and I_2 (bottom disk) are connected by a mass-less torsion spring which produces a restoring torque with magnitude $|\tau|=c|\theta_1-\theta_2|$on each disk, where θ_1 and θ_2 are the angular positions of the disks. (You can imagine that this system is free floating in space subject to no other forces). The disks are held initially at rest with the bottom disk twisted by an angle $\theta_2 = \theta_{2I}$ with respect to equilibrium. The top disk is held initially at $\theta_1=0$.

(a) What is the initial angular momentum? Initially, what is the total mechanical energy including kinetic energy and potential energy?

Initially when release at t = 0, both disks are at rest and hence the angular momentum of the system will be zero.

As the initial angular velocities of the disks zero, initial kinetic energy of the system will be zero.

The potential energy of the system is equal to the work done on the system for angular displacement, which remains stored in the system as elastic potential energy of the system.



Now as in analogy with the tension in a spring is $K\Delta L$ and the energy stored is ½ $K\Delta L^2$ (K is the force constant of the spring and ΔL is its expansion), the torque at an instant in the torsion spring is $C\ \Delta\theta$, and energy stored in the torsion spring is given by ½ $C\ \Delta\theta^2$.

Hence the elastic potential energy stored in the torsion spring will be given by

$$U = \frac{1}{2} C\, \theta_{2i}^2 \qquad \text{---------------- (1)}$$

(b) The disks are then released. What is the maximum rotational kinetic energy acquired by the top disk?

Now when the disks are released, the torque due to torsion on both will act on them in opposite direction and they starts rotate in opposite direction, the elastic potential energy converts in to rotational kinetic energy. When the spring comes to its equilibrium position (angle of torsion is zero), both will have their maximum rotational kinetic energies. The whole potential energy at this time will converts into the kinetic energy.

Let the angular velocities of the disks in equilibrium position are ω_1 and ω_2 respectively.

As there is no external torques acting on the system, those on the disks due to spring is internal and the total angular momentum of the system remains conserved. As the angular momentum is given by the product of moment of inertia and the angular velocity ($I^* \omega$), and the initial angular momentum of the system is zero, we get the final angular momentum of the system

$$I_1 \omega_1 + I_2 \omega_2 = 0$$

Or $$\omega_2 = -\omega_1 \frac{I_1}{I_2} \qquad \text{-------------------------------- (2)}$$

And as there is no resistive non conservative force acting on the system, according to law of conservation of energy

Gain in rotational kinetic energy = loss in elastic potential energy

Or $\frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 = \frac{1}{2}C\,\theta_{2i}^2$

Substituting for ω_2 from equation (2) we get

$$\frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\left(-\omega_1\frac{I_1}{I_2}\right)^2 = \frac{1}{2}C\,\theta_{2i}^2$$

Or $\frac{1}{2}I_1\omega_1^2\left(1+\frac{I_1}{I_2}\right) = \frac{1}{2}C\,\theta_{2i}^2$

Or $\frac{1}{2}I_1\omega_1^2 = \frac{1}{2}\left(\frac{I_2}{I_1+I_2}\right)C\,\theta_{2i}^2$

Hence the maximum angular velocity of the top disk will be $\sqrt{\frac{1}{I_1}\left(\frac{I_2}{I_1+I_2}\right)C\,\theta_{2i}^2}$

(c) What is the oscillation frequency ω of this system?
Let the angular displacement of the disks at time t are θ_1(t) and – θ_2(t) from equilibrium position. The total angle of torsion will be $|\theta_1+\theta_2|$ and hence the restoring torque on the system is given by

$$\tau = -\ C|\theta_1+\theta_2|$$

As the system can be considered as a two body system, its reduced moment of inertia is given by

$$I = \frac{I_1 I_2}{I_1+I_2}$$

Now as the frequency of a body executing rotational simple harmonic motion is given by

$$n = \frac{1}{2\pi}\sqrt{\frac{C}{I}}$$

The frequency of this two body system is given by

$$n = \frac{1}{2\pi}\sqrt{\frac{C(I_1+I_2)}{I_1 I_2}}$$

<u>Alternative method:</u>
Let at time t the angular displacements of the disks are θ_1(t) and θ_2(t) from initial position (θ_1 =0; $\theta_2 = \theta_{2i}$).
The magnitude of the torque due to torsion spring on either disk is

$$\tau = C\ |\theta_2-\theta_1|$$

This torque will be in positive direction on upper disk and in negative direction on lower disk. Thus the equation for rotational motion $\tau = I\beta$ (β is angular acceleration) can be written for the two disks as
For the first (top) disk

$$\tau = I_1 \frac{d^2\theta_1}{dt^2}$$

Or $$\frac{d^2\theta_1}{dt^2} = \frac{C}{I_1} |\theta_2 - \theta_1|$$ ------------------------------------- (3)

Similarly for the second disk it will be

$$\frac{d^2\theta_2}{dt^2} = -\frac{C}{I_2} |\theta_2 - \theta_1|$$ ------------------------------------- (4)

Equation (4) – (3) dives us

$$\frac{d^2\theta_2}{dt^2} - \frac{d^2\theta_1}{dt^2} = -\frac{C}{I_2} |\theta_2 - \theta_1| - \frac{C}{I_1} |\theta_2 - \theta_1|$$

Or $$\frac{d^2(\theta_2 - \theta_1)}{dt^2} = -\left(\frac{C}{I_2} + \frac{C}{I_1}\right) (\theta_2 - \theta_1)$$

Or $$\frac{d^2(\theta_2 - \theta_1)}{dt^2} = -\frac{C(I_1 + I_2)}{I_1 I_2} (\theta_2 - \theta_1)$$

This is similar to the equation of simple harmonic motion

$$a = -\frac{K}{m} x$$

Where k/m is denoted by ω^2 and ω is called angular frequency of the SHM. Or for rotational SHM the equation is given by

$$\beta = -\frac{C}{I} \theta$$

Here $\sqrt{\frac{C}{I}}$ is called the angular frequency of the RSHM.

Thus for our system the motion is RSHM in $(\theta_2 - \theta_1)$, and the angular frequency is given by

$$\omega^2 = \frac{C(I_1 + I_2)}{I_1 I_2}$$

Or $$\omega = \sqrt{\frac{C(I_1 + I_2)}{I_1 I_2}}$$

Hence the frequency of oscillation is given by

$$n = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{C(I_1 + I_2)}{I_1 I_2}}$$

[As the system behaves as a single body with moment of inertia $I = \frac{I_1 I_2}{I_1 + I_2}$
This is called reduced moment of inertia of the system]